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Crum transformation and rational solutions of the non-focusing nonlinear Schrödinger equation

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Abstract. A Crum transformation for the linear problem of the non-focusing nonlinear Schrödinger (NLS) equation is used to construct the sequence of rational solutions, given in terms of tau-functions expressed as double Wronskian determinants. The rational solutions of the Adler–Kaup–Newell–Segur (AKNS) hierarchy have already been obtained in the work of Sachs, using the equivalence with the classical Boussinesq hierarchy. However, we find that Sachs' description of the restrictions that must be placed on the AKNS tau-functions in order to obtain NLS rational solutions is not entirely correct. We briefly comment on the constrained Calogero–Moser systems associated with the NLS rational solutions, and indicate how the Calogero–Moser equations can be obtained from a trilinear equation for the tau-function which arises by reduction of the Kadomtsev–Petviashvili (KP) hierarchy.

1. Introduction

There has been a great deal of interest in the rational solutions of integrable nonlinear evolution equations over the past 20 years. This began with studies of the rational solutions of the Korteweg–de Vries (KdV) [1, 2] and KP [12, 21] equations, but soon similar results were obtained concerning the Benjamin–Ono equation [4] and the classical Boussinesq and AKNS systems [8, 18, 19], among others. An important feature of these rational solutions, which has been further explored more recently [11, 20], is that the motion of their poles is governed by Calogero–Moser systems (possibly with constraints [17]).

In the following we present a construction of rational solutions of the NLS equation

$$\mathrm{i}\psi_t + \Psi_{xx} + 2\delta|\psi|^2\psi = 0 \tag{1.1}$$

in the non-focusing case $\delta = -1$. The focusing case $\delta = +1$ does not admit rational solutions, as is explained in section 2.

Our reasons for considering such solutions are threefold. First, the original motivation behind this work was to use a Crum transformation for the NLS equation (section 3) to generate the sequence of rational solutions (section 4), in complete analogy to Adler and Moser's construction [1] for the KdV equation. Second, although rational solutions of the NLS equation have been considered before, there are certain errors and omissions in the literature. Hirota and Nakamura [8], found a sequence of rational solutions to the non-focusing NLS equation by exploiting a connection with a bilinear Bäcklund transformation (BT) for the classical Boussinesq (cB) system,

$$u_t = ((1+u)v - v_{xx})_x$$

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$$v_t = (u + \tfrac{1}{2}v^2)_x \tag{1.2}$$

where only rational similarity solutions (depending just on x, t) were considered. Sachs gave an exhaustive description of the rational solutions of the cB hierarchy in [18], and also described (with more details in [19]) the rational solutions of the AKNS hierarchy, by using the equivalence of these two hierarchies. Some of these AKNS rational solutions reduce to give rational solutions of the NLS hierarchy. However, contrary to the statement in [19], we find that only the non-focusing NLS hierarchy has rational solutions (section 2), and these depend on all the times of the hierarchy. Our third reason for considering these rational solutions was to see how they fit into the framework of the NLS as a reduction of the KP hierarchy [14]. In this picture, the rational solutions are characterized by a single polynomial tau-function satisfying the trilinear equation of [6], and the zeros of this tau-function satisfy constrained Calogero–Moser systems (briefly considered in section 5).

2. Reduction from AKNS

2.1. AKNS and NLS

In the form (1.1), the equation NLS is really two different equations describing different physical behaviours: the focusing and non-focusing NLS equations, corresponding to $\delta = +1$ and $\delta = -1$, respectively. Both these cases may be obtained from the AKNS system

$$q_{t_2} = q_{xx} + 2q^2 r \qquad r_{t_2} = -r_{xx} - 2qr^2 \tag{2.1}$$

upon setting $t_2 = \mathrm{i}t$, $q = \psi$ and $r = \delta\overline{\psi}$ (for real x, t). (Throughout we use $\overline{Q}$ to denote the complex conjugate of a quantity Q, unless it is explicitly stated that Q and $\overline{Q}$ are independent.) Thus the two different NLS equations give solutions to the AKNS system with particular reality conditions. In [16], Previato derived the hyperelliptic quasiperiodic solutions of the AKNS system using algebraic geometry, and then studied the reality conditions corresponding to $\delta = \pm 1$, showing how certain limits of these solutions gave the N-soliton formulae found by Hirota [7] using his bilinear formalism.

The solutions of the focusing and non-focusing NLS equations are of a very different character. For example, only the non-focusing case of (1.1) has scaling similarity solutions expressible in terms of the fourth Painlevé transcendent [3]. This difference can be seen directly from Painlevé analysis [5, 13]. Expanding around a singular manifold $\phi(x, t)$ in (1.1), it is found that ψ must take the form

$$\psi(x, t) = \phi^{-1} \sum_{n=0}^{\infty} u_n(x, t)\phi^n \tag{2.2}$$

where the leading term satisfies

$$|u_0|^2 = -\delta\phi_x^2.$$

Clearly only the non-focusing case can have a real singular manifold function ϕ. To show that rational solutions may only occur in this case, we now consider the Hirota bilinear form of the NLS equation.

Remark. The Crum transformation described in section 3 may be obtained by truncating the Painlevé expansion (2.2) to give $\psi = u_0/\phi + u_1$, which should be compared with equation (3.6) below. There are many intimate connections between BTs, truncated Painlevé expansions and Hirota bilinear equations (see [5, 10]).

2.2. Bilinear form and polynomial tau-functions

Our construction of rational solutions in section 4 is greatly facilitated by the introduction of the bilinear form of the NLS equation. The dependent variable ψ is given as a ratio of two tau-functions,

$$\psi = \frac{g}{f} \tag{2.3}$$

with f real, and then the NLS equation (1.1) may be separated into a pair of bilinear equations,

$$(\mathrm{i}D_t + D_x^2)g \cdot f = 0 \tag{2.4}$$

$$D_x^2 f \cdot f - 2\delta|g|^2 = 0. \tag{2.5}$$

Up to rescalings of the times τ_j, the higher equations in the NLS hierarchy [13] can be written in bilinear form as

$$(\mathrm{i}D_{\tau_{j+1}} + D_{\tau_j} D_x)g \cdot f = 0. \tag{2.6}$$

This includes (2.4) with the identifications $x = \tau_1$, $t = \tau_2$. Clearly the complex conjugate of (2.6) must also hold (for real τ_j), and then the bilinear equations of the AKNS hierarchy arise when g and its conjugate $\bar{g}$ are regarded as independent tau-functions.

To describe the rational solutions we find it helpful to denote the Wronskian of n functions $a_1, a_2, \ldots, a_n$ by

$$[a_1, a_2, \ldots, a_n] := \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_{1,x} & a_{2,x} & \cdots & a_{n,x} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,(n-1)x} & a_{2,(n-1)x} & \cdots & a_{n,(n-1)x} \end{vmatrix}.$$

We also require the sequence of Schur polynomials p_j for $j = 0, 1, 2, \ldots,$ defined by

$$\exp[\xi(\underline{t}, \nu)] = \sum_{j=0}^{\infty} p_j(\underline{t})\nu^j \qquad \xi(\underline{t}, \nu) = \sum_{j=1}^{\infty} t_J \nu^j \qquad t_1 = x.$$

From this definition it is simple to show the following:

$$\frac{\partial^k p_j}{\partial x^k} = p_{j-k} = \frac{\partial p_j}{\partial t_k}. \tag{2.7}$$

In [8] Hirota and Nakamura used the substitutions

$$u = -1 - 2(\log[F\overline{F}])_{xx} \qquad v = 2\mathrm{i}(\log[F/\overline{F}])_x$$

writing the cB system (1.2) in bilinear form:

$$(\mathrm{i}D_t + D_x^2)\overline{F} \cdot F = 0 \qquad (\mathrm{i}D_t D_x + D_x^3)\overline{F} \cdot F = 0.$$

Hirota and Nakamura gave a bilinear Bäcklund transformation (BT) for this conjugate pair of tau-functions $F, \overline{F}$, and proceeded to show that given another pair $F', \overline{F}'$ related by the BT, tau-functions for the non-focusing NLS equation may be constructed from the formulae

$$f^2 = \tfrac{1}{2}(F\overline{F}' + F'\overline{F}) \qquad gf = \tfrac{1}{2}(D_x \overline{F} \cdot \overline{F}'). \tag{2.8}$$

The bilinear BT is used in [8] to construct a sequence of rational solutions ('explode-decay' solitons) to (1.2), and it is proved in [9] that the polynomial tau-functions $F, \overline{F}$ may be written as Wronskians of Hermite polynomials in the similarity variable $z = x/t^{\frac{1}{2}}$. However, it is still not apparent from the substitutions (2.8) that the associated NLS tau-functions are polynomials. In section 4 we show that in fact the expressions (2.8) arise naturally from the Crum transformation.

In the work of Sachs, the cB hierarchy is considered without the particular reality conditions taken in [8] regarding $F, \overline{F}$ as independent tau-functions. It was shown in [18] that the general rational solution of the cB hierarchy has tau-functions of the form

$$F = [p_n, p_{n-1}, \ldots, p_{n-k}] \qquad \overline{F} = [p_n, p_{n-1}, \ldots, p_{n-k+1}]$$

where k and n are integers with $0 \leqslant k \leqslant n$, and $x = t_1, t_2, t_3, \ldots$ are the times of the hierarchy. The scaling similarity solutions of [8] correspond to the particular choice $n = 2k$, upon setting $t_2 = \mathrm{i}t$ and $t_3 = t_4 \cdots = 0$. Also, using the fact that the cB and AKNS hierarchies are equivalent, it was shown [18, 19] that the most general rational solution of the AKNS hierarchy has tau-functions in the form of double Wronskians,

$$f = [p_n, \ldots, p_{n-k}] \qquad g = [p_n, \ldots, p_{n-k-1}] \qquad \overline{g} = [p_n, p_{n-1}, \ldots, p_{n-k+1}] \tag{2.9}$$

(regarding g and $\overline{g}$ as independent). Making the reduction to the NLS hierarchy requires that f must be real and g and $\overline{g}$ must be complex conjugates (and thus in particular they must be of the same degree in x). A simple check of the degrees of the Wronskians (2.9) shows that for this reduction it is necessary to take $n = 2k + 1$, in agreement with [19]. However, it is further stated in [19] that this reduction is consistent for both values $\delta = \pm 1$, and that for consistency it is necessary to set the odd times to zero ($t_3 = t_5 = \cdots = 0$), which is not correct.

To see why the focusing NLS equation cannot have rational solutions, one need only consider the bilinear equation (2.5), which (from (2.3)) clearly entails

$$|\psi|^2 = \delta(\log[f])_{xx}. \tag{2.10}$$

Note that in (2.3) there is always the freedom to rescale f and g by the same arbitrary constant. Thus, without loss of generality, when considering rational solutions of the NLS equation we may take f to be a monic polynomial in x, so that

$$f = \prod_{j=1}^{N}(x - x_j(\underline{t})) \qquad g = \prod_{J=1}^{M}(x - y_J(\underline{t})) \tag{2.11}$$

for some constant κ (with $\underline{t}$ denoting the sequence of times of the hierarchy). Substituting these polynomials into (2.10) and multiplying through by the common denominator f^2 yields

$$|\kappa|^2 \prod_{J=1}^{M}(x - y_J)(x - \overline{y_J}) = -\delta \sum_{j=1}^{N} \prod_{k \neq j}(x - x_k)^2. \tag{2.12}$$

Comparing the terms of highest degree in x on either side of (2.12) implies immediately that $M = N - 1$ and that $|\kappa|^2 = -\delta N$, forcing $\delta = -1$. The condition $M = N - 1$ can be checked directly for the Wronskians (2.9) with $n = 2k + 1$, and it is found that $N = (k+1)^2$. The construction presented in section 4 provides a direct proof that these Wronskians satisfy the necessary reality conditions for the non-focusing NLS equation, and that it is consistent to include all the flows of the hierarchy.

3. NLS Crum transformation

The non-focusing NLS equation arises as the zero curvature condition for the linear system

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_x = \begin{pmatrix} -\mathrm{i}\lambda & \psi \\ \overline{\psi} & \mathrm{i}\lambda \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{3.1}$$

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_t = \begin{pmatrix} -\mathrm{i}(|\psi|^2 + 2\lambda^2) & \mathrm{i}\psi_x + 2\lambda\psi \\ -\mathrm{i}\overline{\psi}_x + 2\lambda\overline{\psi} & \mathrm{i}(|\psi|^2 + 2\lambda^2) \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{3.2}$$

Henceforth we shall only consider the x part (3.1), and make the consistent choice

$$\chi_1 = \chi \qquad \chi_2 = \overline{\chi}$$

leading to

$$\chi_x = -\mathrm{i}\lambda\chi + \psi\overline{\chi} \tag{3.3}$$

for λ real. By a slight abuse of language, we shall refer to χ as the 'eigenfunction' and ψ as the 'potential' in (3.3).

To describe the Crum transformation for the system (3.3) requires a real singular manifold function ϕ, obtained via

$$\phi_x = |\chi|^2. \tag{3.4}$$

Now we can define a new eigenfunction

$$\chi^* = \frac{\chi}{\phi}.$$

It is easy to see that $X = \chi^*$ is a solution to the equation

$$X_x = -\mathrm{i}\lambda X + \tilde{\psi}\overline{X} \tag{3.5}$$

with the new potential

$$\tilde{\psi} = \psi - \frac{\chi^2}{\phi}. \tag{3.6}$$

Equation (3.6) may be regarded as a truncation of the expansion (2.2).

Having obtained $\tilde{\psi}$ from ψ by one application of the Crum transformation for (3.3), a sequence of such potentials may be obtained by applying it repeatedly. A new singular manifold function ϕ^* can be found from $\phi^*_x = |v^*|^2$, but up to a constant we must have $\phi^* = -\phi^{-1}$, and so applying the same transformation again just leads back to the original potential ψ. Thus to get anything new requires a new eigenfunction χ' satisfying (3.5), such that χ' and χ^* are linearly independent (over the reals). It may be checked that the determinant

$$\tilde{W}[\chi', \chi^*] := (2\mathrm{i})^{-1} \begin{vmatrix} \chi' & \chi^* \\ \overline{\chi}' & \overline{\chi}^* \end{vmatrix} \tag{3.7}$$

is constant for any two solutions of (3.5), and non-zero when they are independent. In the next section we apply the Crum transformation to obtain the sequence of rational solutions of the non-focusing NLS equation, and use the quantity $\tilde{W}$ to normalize the eigenfunctions in a particular way.

Remark. A similar Crum transformation has been used to generate soliton solutions of the Davey–Stewartson equation in [22].

4. Sequence of rational solutions

4.1. Application of Crum transformation

We proceed to construct a sequence of rational functions ψ_k for $k = 0, 1, 2, \ldots$ by repeated application of the Crum transformation with the eigenvalue $\lambda = 0$, starting from the vacuum $\psi_0 = 0$. Carrying out all the manipulations in bilinear form leads to the following.

Proposition 1. Repeated application of the Crum transformation for (3.5) with eigenvalue $\lambda = 0$, starting from the vacuum potential $\psi_0 = 0$, is equivalent to repeatedly solving the bilinear equations

$$D_x h_k \cdot f_k = \overline{h}_k g_k \tag{4.1}$$

$$h_k \overline{h}_{k-1} - \overline{h}_k h_{k-1} = 2\mathrm{i}\sqrt{(2k-1)(2k+1)}\, f_k^2 \tag{4.2}$$

$$D_x f_{k+1} \cdot f_k = |h_k|^2 \tag{4.3}$$

$$g_{k+1} f_k - g_k f_{k+1} = -h_k^2 \tag{4.4}$$

starting from the initial conditions $f_0 = 1$, $h_0 = \mathrm{i}$. The sequence of potentials ψ_k, the singular manifold functions ϕ_k and the pairs of eigenfunctions χ_k^*, χ_k are given by

$$\psi_k = \frac{g_k}{f_k} \qquad \phi_k = \frac{f_{k+1}}{f_k} \qquad \chi_k^* = \frac{h_{k-1}}{f_k} \qquad \chi_k = \frac{h_k}{f_k} \tag{4.5}$$

where

$$|\psi_k|^2 = -(\log[f_k])_{xx}. \tag{4.6}$$

Proof. First we consider $k = 0$. Setting $f_0 = 1, h_0 = \mathrm{i}$ in (4.1) gives $g_0 = 0$, so that $\psi_0 = 0$. Integrating (3.4) with $\chi_0 = h_0/f_0$ gives $\phi_0 = f_1/f_0 = x + \tau_1$, with τ_1 constant, and this is the same as solving (4.3) for f_1. As usual we neglect the translation in x and take $f_1 = x$. From (3.6) or from (4.4) the new potential is found to be $\psi_1 = 1/x$, with $g_1 = 1$. The proof proceeds by a simple induction. With the substitutions (4.5), it is clear that (4.1) is equivalent to $\chi = \chi_k$ being an eigenfunction for the linear problem (3.5) with potential $\psi = \psi_k$ and eigenvalue $\lambda = 0$. Similarly (4.3) is equivalent to (3.4), and (4.4) is equivalent to (3.6). It is also straightforward to show (using (4.1), (4.3) and (4.4) with $k \to k-1$) that h_{k-1} satisfies

$$D_x h_{k-1} \cdot f_k - \overline{h}_{k-1} g_k \tag{4.7}$$

which implies that χ_k^* is another eigenfunction for (3.5) with potential ψ_k; comparison with (3.7) shows that (4.2) is just the normalization condition $\tilde{W}[\chi_k, \chi_k^*] = \sqrt{(2k-1)(2k+1)}$ for the independent pair of eigenfunctions. Equation (4.6), which is equivalent to the bilinear (2.5) for $\delta = -1$, also follows inductively by calculating expressions for $\phi_{k,x}$ and $\phi_{k,xx}$. □

Implementing the Crum transformation algorithmically requires two integrations and solving one algebraic equation at each step. Indeed, a consequence of (4.1), (4.2) and (4.7) is the bilinear

$$D_x h_k \cdot h_{k-1} = -2\mathrm{i}\sqrt{(2k+1)(2k-1)}\, f_k g_k. \tag{4.8}$$

It follows that given h_{k-1}, f_k, g_k found by k applications of the Crum transformation in the form (4.1)–(4.4), the new tau-function h_k may be obtained by integrating

$$\left(\frac{h_k}{h_{k-1}}\right)_x = -2\mathrm{i}\sqrt{(2k+1)(2k-1)}\,\frac{f_k g_k}{h_{k-1}^2}.$$

The constant of integration, which we denote $\sqrt{(2k+1)(2k-1)}\tau_{2k}$, is just the arbitrary real multiple of h_{k-1} that may be added to h_k. Next, (using (4.3) or (3.4)) the tau-function f_{k+1} is found by integrating

$$\left(\frac{f_{k+1}}{f_k}\right)_x = \frac{|h_k|^2}{f_k^2}.$$

Similarly, an arbitrary real multiple of f_k can be added to f_{k+1}, and we denote this second constant of integration by τ_{2k+1}. Finally, (4.4) is rearranged to solve for g_{k+1}. As is the case for the KdV equation [1], when applying the Crum transformation for the rational solutions the constants of integration τ_j correspond to the times of the hierarchy. A slightly tedious direct calculation shows that for each new pair of times τ_{2k}, τ_{2k+1} we have

$$(2\mathrm{i}(2k+1)(2k-1)D_{\tau_{2k+1}} + D_x D_{\tau_{2k}})g_{k+1} \cdot f_{k+1} = 0$$

which is the binilear form of the $(2k+1)$th flow of the NLS hierarchy (identical to (2.6) after a rescaling).

We list the first few tau-functions found from the Crum transformation:

$$f_0 = 1 \qquad f_1 = x \qquad f_2 = x^4 + \tau_3 x - 3\tau_2^2$$
$$f_3 = x^9 + 6\tau_3 x^6 - 18\tau_2^2 x^5 + \tau_5 x^4 - 60\tau_2\tau_4 x^3 + 90\tau_2^2\tau_3 x^2 + (\tau_3\tau_5 - 135\tau_2^4 - 15\tau_4^2)x$$
$$+30\tau_2\tau_3\tau_4 - 5\tau_3^2 - 3\tau_2^2\tau_5$$
$$g_0 = 0 \qquad g_1 = 1 \qquad g_2 = -2x^3 + 6\mathrm{i}\tau_2 x + \tau_3$$
$$g_3 = 3x^8 - 24\mathrm{i}\tau_2 x^6 + 6\tau_3 x^5 - 30(3\tau_2^2 + \mathrm{i}\tau_4)x^4 - 2(\tau_5 - 30\mathrm{i}\tau_2\tau_3)x^3 + 30\tau_3^2 x^2$$
$$+(-90\tau_2^2\tau_3 + \mathrm{i}(6\tau_2\tau_5 - 30\tau_3\tau_4))x + 135\tau_2^4 + \tau_3\tau_5 - 15\tau_4^2$$
$$+30\mathrm{i}(\tau_2\tau_3^2 - 3\tau_2^2\tau_4)$$
$$h_0 = \mathrm{i} \qquad h_1 = -\sqrt{3}(x^2 - \mathrm{i}\tau_2)$$
$$h_2 = -\mathrm{i}\sqrt{5}\begin{pmatrix} x^6 - 3\mathrm{i}\tau_2 x^4 + 2\tau_3 x^3 - (9\tau_2^2 + 3\mathrm{i}\tau_4)x^2 \\ +6\mathrm{i}\tau_2\tau_3 x + \tau_3^2 - 3\tau_2\tau_4 - 9\mathrm{i}\tau_2^3 \end{pmatrix}.$$

After setting $\tau_2 = 2t$ and $\tau_j = 0$ for $j \leqslant 3$, these agree with the tau-functions for similarity solutions found in [8]. Note that the particular normalization used here was chosen to make the f_k monic polynomials in x. To prove that all the tau-functions generated by this Crum transformation are polynomials, we must now show that they are proportional to double Wronskians of Schur polynomials (thus making contact with the formulae of Sachs [18]).

Remark. Upon multiplying either h_{n-1} or h_n by i and rescaling, the equations (4.2) and (4.8) are identical to the formulae (2.8) obtained by Hirota and Nakamura [8].

4.2. Wronskian formulae

We now introduce Wronskians of the form (2.9):

$$F_k = [p_{2k-1}, \ldots, p_k] \qquad G_k = [p_{2k-1}, \ldots, p_{k-1}] \qquad \overline{G}_k = [p_{2k-1}, \ldots, p_{k+1}]$$
$$H_k = [p_{2k}, \ldots, p_k] \qquad \overline{H}_k = [p_{2k}, \ldots, p_{k+1}].$$

It will turn out that the complex conjugate of G_k (respectively, H_k) is equal to $\overline{G}_k$ (respectively, $\overline{H}_k$) up to a minus sign, but to see this we must first show the following.

Proposition 2. The double Wronskians satisfy rescaled versions of (4.1)–(4.4), i.e.

$$D_x H_k \cdot F_k = \overline{H}_k G_k \qquad D_x F_{k+1} \cdot F_k = H_k \overline{H}_k \tag{4.9}$$
$$H_k \overline{H}_{k-1} - \overline{H}_k H_{k-1} = -F_k^2 \qquad G_{k+1}F_k - G_k F_{k+1} - = H_k^2. \tag{4.10}$$

The 'conjugate' equations, obtained by swapping G_k with $\overline{G}_k$, and H_k with $\overline{H}_k$ (and inserting a minus sign where necessary), are also satisfied.

Proof. Using the property (2.7) of Schur polynomials, the first equation in (4.9) is equivalent to

$$[p_{2k}, \ldots, p_{k+1}, p_{k-1}] \cdot [p_{2k-1}, \ldots, p_k] - [p_{2k}, \ldots, p_k] \cdot [p_{2k-1}, \ldots, p_{k+1}, p_{k-1}]$$
$$-[p_{2k-1}, \ldots, p_{k-1}] \cdot [p_{2k-1}, \ldots, p_{k+1}] = 0. \tag{4.11}$$

To see why (4.11) must hold, observe that it is just the Laplace expansion in the first $(k+1)$ rows of the determinant

$$\left|\begin{array}{c|ccc|c|c|ccc}
p_{2k} & p_{2k-1} & \ldots & p_{k+1} & p_k & p_{k-1} & & & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & 0 & \\
p_k & p_{k-1} & \cdots & p_1 & p_0 & 0 & & & \\
\hline
p_{2k} & & & & p_k & p_{k-1} & p_{2k-1} & \ldots & p_{k+1} \\
\vdots & & 0 & & \vdots & \vdots & \vdots & \ddots & \vdots \\
p_{k+1} & & & & p_1 & p_0 & p_k & \cdots & p_2
\end{array}\right|. \tag{4.12}$$

It is straightforward to show that (4.12) vanishes, and the Laplace expansion of a similar determinant yields the second equation in (4.9). The first equation in (4.10) follows from the Laplace expansion of the determinant

$$\left|\begin{array}{c|ccc|c|c|ccc}
p_{2k} & p_{2k-1} & \ldots & p_{k+1} & p_k & 0 & & & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & 0 & \\
p_{k+1} & p_k & \cdots & p_2 & p_1 & 0 & & & \\
p_k & p_{k-1} & \cdots & p_1 & p_0 & 1 & & & \\
\hline
p_{2k-1} & & & & p_{k-1} & 0 & p_{2k-2} & \ldots & p_k \\
\vdots & & 0 & & \vdots & \vdots & \vdots & \ddots & \vdots \\
p_{k+1} & & & & p_1 & 0 & p_k & \cdots & p_2 \\
p_k & & & & p_0 & 1 & p_{k-1} & \cdots & p_1
\end{array}\right|$$

and similarly for the second equation in (4.10) and the 'conjugates'. □

An immediate consequence of the above is that the tau-functions found via the Crum transformation (as in proposition 1) must be proportional to the double Wronskians, provided that the constants of integration τ_j can be identified with the t_j. A direct comparison shows that

$$f_k = (-)^{[k/2]}\frac{k!}{(2k)!}c(k)F_k \qquad g_k = (-)^{k+1+[(k+1)/2]}\frac{(2k-1)!}{(k-1)!}c(k-1)G_k$$
$$\overline{g}_k = (-)^{k+1+[(k+1)/2]}\frac{(2k-1)!}{(k-1)!}c(k-1)\overline{G}_k$$
$$h_k = (-)^{[(k+1)/2]}\,\mathrm{i}^{k+1}\sqrt{2k+1}c(k)H_k \qquad \overline{h}_k = (-)^{[k/2]}(-\mathrm{i})^{k+1}\sqrt{2k+1}c(k)\overline{H}_k$$

where

$$c(k) := \prod_{j=0}^{k}\frac{(k+j)!}{j!}$$

and

$$t_{2k} = (-)^{k+1}\frac{(k-1)!k!}{(2k-2)!(2k)!}\mathrm{i}\tau_{2k} \qquad t_{2k+1} = (-)^k\frac{(k!)^2}{(2k)!(2k+1)!}\tau_{2k+1}.$$

It then follows from the general results of [18, 19] that, subject to the above reality conditions on the times t_j, the g_k, f_k generated by the Crum transformation are tau-functions of the (non-focusing) NLS hierarchy, and the $h_k, \overline{h}_k$ are tau-functions of the cB hierarchy.

5. Trilinear form and Calogero–Moser

While solutions of the NLS equation are commonly given in terms of a pair of tau-functions g, f satisfying bilinear equations, if trilinear equations are employed then these solutions can also be (almost completely) characterized by the single tau-function f. Starting from the coupled equations for the modulus-squared w and phase γ of ψ (i.e. writing $\psi = w^{\frac{1}{2}} \exp[\mathrm{i}\gamma]$), and introducing the variable $\eta = -2w\gamma_x$, the NLS equation (1.1) (taking $\delta = -1$) leads to the system

$$w_t = \eta_x \tag{5.1}$$

$$\eta_t = \left(2w^2 - w_{xx} + \frac{w_x^2 + \eta^2}{w}\right)_x . \tag{5.2}$$

Note that the phase γ is only determined by η up to a function of t. Defining $\Lambda = \log[f]$, we see that (4.6) is equivalent to

$$w = -\Lambda_{xx}$$

so that we may integrate first (5.1) to find

$$\eta = -\Lambda_{xt}$$

and then (5.2) to obtain

$$\Lambda_{tt}\Lambda_{xx} - \Lambda_{xt}^2 - \Lambda_{3x}^2 + 2\Lambda_{xx}^3 + \Lambda_{xx}\Lambda_{4x} = 0. \tag{5.3}$$

Both arbitrary functions of t occurring in the integration are consistently absorbed into f. Multiplying through by f^3 in (5.3) yields the trilinear form

$$\begin{vmatrix} f & f_x & f_t \\ f_x & f_{xx} & f_{xt} \\ f_t & f_{xt} & f_{tt} \end{vmatrix} + \begin{vmatrix} f & f_x & f_{xx} \\ f_x & f_{xx} & f_{3x} \\ f_{xx} & f_{3x} & f_{4x} \end{vmatrix} = \begin{vmatrix} p_0^+ p_0^-(\tau) & p_0^+ p_1^-(\tau) & p_0^+ p_2^-(\tau) \\ p_1^+ p_0^-(\tau) & p_1^+ p_1^-(\tau) & p_1^+ p_2^-(\tau) \\ p_2^+ p_0^-(\tau) & p_2^+ p_1^-(\tau) & p_2^+ p_2^-(\tau) \end{vmatrix} = 0. \tag{5.4}$$

This is the trilinear form of the Kaup–Broer system [6], which is equivalent to the AKNS system (2.1). The second expression in (5.4) is the form of the equation used in [14], with the operators $p_j^{\pm} = p_j(\pm\tilde{\partial})$, $\tilde{\partial} = (\partial_x, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \ldots)$; it is equal to the first expression when we identify $\tau = f, t_2 = \mathrm{i}t$. It is shown in [14] that the AKNS hierarchy arises by constraining the Lax operator of the KP hierarchy to be

$$L = \partial_x + q\partial_x^{-1} r$$

where q, r are the wavefunction and adjoint wavefunction, respectively. This is known as the generalized 1-constraint, and puts trilinear constraints on the KP tau-function τ.

The bilinear KP hierarchy has solutions given by tau-functions expressed as Wronskian determinants [15]. Double Wronskians of a particular form [6] give solutions to (5.4); the polynomial tau-functions we have considered are particular examples of these. Rational solutions of the KP hierarchy are associated with Calogero–Moser particle systems [20], and thus the trilinear equation puts constraints on these systems. Substituting a polynomial tau-function f (2.11) into (5.3) yields a partial fraction expansion around the poles $x = x_j$. The coefficients of $(x - x_j)^{-3}$ give the Calogero–Moser equations

$$\ddot{x}_j = 8\sum_{k\neq j}(x_j - x_k)^{-3} \tag{5.5}$$

(with dot denoting ∂_t). The coefficients of the $(x - x_j)^{-2}$ and $(x - x_j)^{-1}$ terms lead to complicated constraints [10] on the first derivatives $\dot{x}_j$; a natural interpretation for these constraints is lacking, but we intend to consider them elsewhere. Alternatively, substituting the polynomials (2.11) into the bilinear equation (2.4) leads to a coupled first-order system for the poles x_j and zeros y_J of ψ, further constrained by (2.12). It is straightforward to show that this first-order system decouples into two Calogero–Moser systems, i.e. (5.5) and

$$\ddot{y}_J = 8 \sum_{K \neq J} (y_J - y_K)^{-3}.$$

There are similar constrained, coupled Calogero–Moser systems associated with the rational solutions of the cB hierarchy [17, 18].

We have also observed [10] that some of the similarity solutions of the NLS equation considered in [3] correspond to solutions of (5.4) of the form

$$f = \exp[p(x, \underline{t})] \prod_{j=1}^{N} (x - x_j(\underline{t}))$$

where p is a quartic polynomial in x. Tau-functions of this form yield the non-decreasing rational solutions of KP considered by Veselov [21], and are also associated with systems of Calogero–Moser type.

6. Conclusions

We have shown how a Crum transformation may be used to construct the rational solutions of the non-focusing NLS hierarchy, in analogy with the construction used by Adler and Moser [1] for the KdV equation. At the same time, we have found a natural derivation of the formulae (2.8) found by Hirota and Nakamura in connection with a bilinear BT [8] for the cB system (1.2). In applying our construction, we have also noted that Sachs' description [19] of the reduction from AKNS rational solutions contains two errors: first, the focusing NLS equation does not admit rational solutions; and second, it is not necessary to set the odd times $t_3, t_5, \ldots$ to zero. We have also briefly considered the constrained Calogero–Moser systems associated with the poles and zeros of the NLS rational solutions. In particular, the pole motion is characterized by a trilinear equation (5.4) arising by reduction of the KP hierarchy; we hope to explore this further elsewhere.

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